9-5

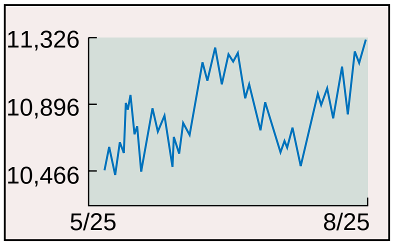
Simplifying Complex Numbers

OBJECTIVE

- Add, subtract, multiply, and divide complex numbers in rectangular form.

Real World Application

DYNAMICAL SYSTEMS Dynamical systems is a branch of mathematics that studies constantly changing systems like the stock market, the weather, and population. In many cases, one can catch a glimpse of the system at some point in time, but the forces that act on the system cause it to change quickly. By analyzing how a dynamical system changes over time, it may be possible to predict the behavior of the system in the future. One of the basic mathematical models of a dynamical system is iteration of a complex function. *A problem related to this will be solved in Example 4.*



A Typical Graph of the Stock Market

Recall that complex numbers are numbers of the form $a + bi$, where a and b are real numbers and $\boldsymbol{i}$, the imaginary unit, is defined by $i^2 = -1$. The first few powers of $\boldsymbol{i}$ are shown below.

$i^1 = i$	$i^2 = -1$	$i^3 = i^2 \cdot i = -i$	$i^4 = (i^2)^2 = 1$
$i^5 = i^4 \cdot i = i$	$i^6 = i^4 \cdot i^2 = -1$	$i^7 = i^4 \cdot i^3 = -i$	$i^8 = (i^2)^4 = 1$

Notice the repeating pattern of the powers of $\boldsymbol{i}$.

$$i, -1, -i, 1, i, -1, -i, 1$$

In general, the value of i^n, where n is a whole number, can be found by dividing n by 4 and examining the remainder as summarized in the table at the right.

To find the value of i^n, let R be the remainder when n is divided by 4.	
if R = 0	$i^n = 1$
if R = 1	$i^n = i$
if R = 2	$i^n = -1$
if R = 3	$i^n = -i$

You can also simplify any integral power of $\boldsymbol{i}$ by rewriting the exponent as a multiple of 4 plus a positive remainder.

Example 1 **Simplify each power of i.**

a. i^{53}

Method 1
$53 \div 4 = 13$ R1
If R = 1, $i^n = i$.
$i^{53} = i$

Method 2
$i^{53} = (i^4)^{13} \cdot i$
$= (1)^{13} \cdot i$
$= i$

b. i^{-13}

Method 1
$-13 \div 4 = -4$ R3
If R = 3, $i^n = -i$.
$i^{-13} = -i$

Method 2
$i^{-13} = (i^4)^{-4} \cdot i^3$
$= (1)^{-4} \cdot i^3$
$= -i$

The complex number $a + bi$, where a and b are real numbers, is said to be in **rectangular form.** a is called the **real part** and b is called the **imaginary part.** If $b = 0$, the complex number is a real number. If $b \neq 0$, the complex number is an **imaginary number.** If $a = 0$ and $b \neq 0$, as in $4\boldsymbol{i}$, then the complex number is a **pure imaginary number.** Complex numbers can be added and subtracted by performing the chosen operation on both the real and imaginary parts.

Example 2 **Simplify each expression.**

Graphing Calculator Tip

Some calculators have a complex number mode. In this mode, they can perform complex number arithmetic.

a. $(5 - 3i) + (-2 + 4i)$

$$(5 - 3\boldsymbol{i}) + (-2 + 4\boldsymbol{i}) = [5 + (-2)] + [-3\boldsymbol{i} + 4\boldsymbol{i}]$$
$$= 3 + \boldsymbol{i}$$

b. $(10 - 2i) - (14 - 6i)$

$$(10 - 2\boldsymbol{i}) - (14 - 6\boldsymbol{i}) = 10 - 2\boldsymbol{i} - 14 + 6\boldsymbol{i}$$
$$= -4 + 4\boldsymbol{i}$$

The product of two or more complex numbers can be found using the same procedures you use when multiplying binomials.

Example 3 **Simplify $(2 - 3i)(7 - 4i)$.**

$$\begin{aligned}(2 - 3\boldsymbol{i})(7 - 4\boldsymbol{i}) &= 7(2 - 3\boldsymbol{i}) - 4\boldsymbol{i}(2 - 3\boldsymbol{i}) && \textit{Distributive property}\\ &= 14 - 21\boldsymbol{i} - 8\boldsymbol{i} + 12\boldsymbol{i}^2 && \textit{Distributive property}\\ &= 14 - 21\boldsymbol{i} - 8\boldsymbol{i} + 12(-1) && \boldsymbol{i}^2 = -1\\ &= 2 - 29\boldsymbol{i}\end{aligned}$$

Iteration is the process of repeatedly applying a function to the output produced by the previous input. When using complex numbers with functions, it is traditional to use z for the independent variable.

Example 4 Real World Application

DYNAMICAL SYSTEMS **If $f(z) = (0.5 + 0.5i)z$, find the first five iterates of f for the initial value $z_0 = 1 + i$. Describe any pattern that you see.**

$$f(z) = (0.5 + 0.5\boldsymbol{i})z$$

$$\begin{aligned}f(1 + \boldsymbol{i}) &= (0.5 + 0.5\boldsymbol{i})(1 + \boldsymbol{i}) && \textit{Replace } z \textit{ with } 1 + \boldsymbol{i}.\\ &= 0.5 + 0.5\boldsymbol{i} + 0.5\boldsymbol{i} + 0.5\boldsymbol{i}^2\\ &= \boldsymbol{i} && z_1 = \boldsymbol{i}\end{aligned}$$

$$\begin{aligned}f(\boldsymbol{i}) &= (0.5 + 0.5\boldsymbol{i})\boldsymbol{i}\\ &= 0.5\boldsymbol{i} + 0.5\boldsymbol{i}^2\\ &= -0.5 + 0.5\boldsymbol{i} && z_2 = -0.5 + 0.5\boldsymbol{i}\end{aligned}$$

$$\begin{aligned}f(-0.5 + 0.5\boldsymbol{i}) &= (0.5 + 0.5\boldsymbol{i})(-0.5 + 0.5\boldsymbol{i})\\ &= -0.25 + 0.25\boldsymbol{i} - 0.25\boldsymbol{i} + 0.25\boldsymbol{i}^2\\ &= -0.5 && z_3 = -0.5\end{aligned}$$

(continued on the next page)

interNET CONNECTION

Graphing Calculator Programs
To download a graphing calculator program that performs complex iteration, visit: **www.amc.glencoe.com**

$$f(-0.5) = (0.5 + 0.5\boldsymbol{i})(-0.5)$$
$$= -0.25 - 0.25\boldsymbol{i} \qquad z_4 = -0.25 - 0.25\boldsymbol{i}$$

$$f(-0.25 - 0.25\boldsymbol{i}) = (0.5 + 0.5\boldsymbol{i})(-0.25 - 0.25\boldsymbol{i})$$
$$= -0.125 - 0.125\boldsymbol{i} - 0.125\boldsymbol{i} - 0.125\boldsymbol{i}^2$$
$$= -0.25\boldsymbol{i} \qquad z_5 = -0.25\boldsymbol{i}$$

The first five iterates of $1 + \boldsymbol{i}$ are $\boldsymbol{i}$, $-0.5 + 0.5\boldsymbol{i}$, -0.5, $-0.25 - 0.25\boldsymbol{i}$, and $-0.25\boldsymbol{i}$. The absolute values of the nonzero real and imaginary parts (1, 0.5, 0.25) stay the same for two steps and then are halved.

Two complex numbers of the form $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are called **complex conjugates.** Recall that if a quadratic equation with real coefficients has complex solutions, then those solutions are complex conjugates. Complex conjugates also play a useful role in the division of complex numbers. To simplify the quotient of two complex numbers, multiply the numerator and denominator by the conjugate of the denominator. The process is similar to rationalizing the denominator in an expression like $\frac{1}{3 + \sqrt{2}}$.

Example 5 **Simplify $(5 - 3i) \div (1 - 2i)$.**

$$(5 - 3\boldsymbol{i}) \div (1 - 2\boldsymbol{i}) = \frac{5 - 3\boldsymbol{i}}{1 - 2\boldsymbol{i}}$$
$$= \frac{5 - 3\boldsymbol{i}}{1 - 2\boldsymbol{i}} \cdot \frac{1 + 2\boldsymbol{i}}{1 + 2\boldsymbol{i}} \qquad \textit{Multiply by 1; } 1 + 2\boldsymbol{i} \textit{ is the conjugate of } 1 - 2\boldsymbol{i}.$$
$$= \frac{5 + 10\boldsymbol{i} - 3\boldsymbol{i} - 6\boldsymbol{i}^2}{1 - 4\boldsymbol{i}^2}$$
$$= \frac{5 + 7\boldsymbol{i} - 6(-1)}{1 - (-4)} \qquad \boldsymbol{i}^2 = -1$$
$$= \frac{11 + 7\boldsymbol{i}}{5}$$
$$= \frac{11}{5} + \frac{7}{5}\boldsymbol{i} \qquad \textit{Write the answer in the form } a + b\boldsymbol{i}.$$

The list below summarizes the operations with complex numbers presented in this lesson.

Operations with Complex Numbers

For any complex numbers $a + bi$ and $c + di$, the following are true.

$$(a + bi) + (c + di) = (a + c) + (b + d)\boldsymbol{i}$$
$$(a + bi) - (c + di) = (a - c) + (b - d)\boldsymbol{i}$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)\boldsymbol{i}$$
$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\boldsymbol{i}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how to simplify any integral power of i.
2. **Draw** a Venn diagram to show the relationship between real, pure imaginary, and complex numbers.
3. **Explain** why it is useful to multiply by the conjugate of the denominator over itself when simplifying a fraction containing complex numbers.
4. **Write** a quadratic equation that has two complex conjugate solutions.

Guided Practice

Simplify.

5. i^{-6}
6. $i^{10} + i^2$
7. $(2 + 3i) + (-6 + i)$
8. $(2.3 + 4.1i) - (-1.2 - 6.3i)$
9. $(2 + 4i) + (-1 + 5i)$
10. $(-2 - i)^2$
11. $\dfrac{i}{1 + 2i}$

Look Back
You can refer to Lessons 8-1 and 8-2 to review vectors, components, and resultants.

12. **Vectors** It is sometimes convenient to use complex numbers to represent vectors. A vector with a horizontal component of magnitude a and a vertical component of magnitude b can be represented by the complex number $a + bi$. If an object experiences a force with a horizontal component of 2.5 N and a vertical component of 3.1 N as well as a second force with a horizontal component of −6.2 N and a vertical component of 4.3 N, find the resultant force on the object. Write your answer as a complex number.

EXERCISES

Practice

Simplify.

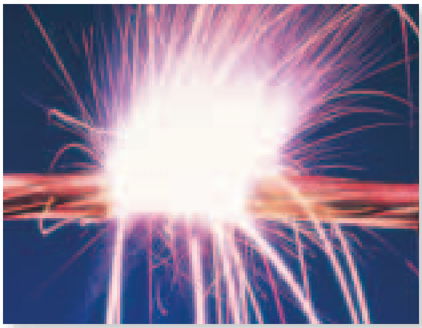
13. i^6
14. i^{19}
15. i^{1776}
16. $i^9 + i^{-5}$
17. $(3 + 2i) + (-4 + 6i)$
18. $(7 - 4i) + (2 - 3i)$
19. $\left(\frac{1}{2} + i\right) - (2 - i)$
20. $(-3 - i) - (4 - 5i)$
21. $(2 + i)(4 + 3i)$
22. $(1 + 4i)^2$
23. $\left(1 + \sqrt{7}i\right)\left(-2 - \sqrt{5}i\right)$
24. $\left(2 + \sqrt{-3}\right)\left(-1 + \sqrt{-12}\right)$
25. $\dfrac{2 + i}{1 + 2i}$
26. $\dfrac{3 - 2i}{-4 - i}$
27. $\dfrac{5 - i}{5 + i}$
28. Write a quadratic equation with solutions i and $-i$.
29. Write a quadratic equation with solutions $2 + i$ and $2 - i$.

Simplify.

30. $(2 - i)(3 + 2i)(1 - 4i)$
31. $(-1 - 3i)(2 + 2i)(1 - 2i)$
32. $\dfrac{\frac{1}{2} + \sqrt{3}i}{1 - \sqrt{2}i}$
33. $\dfrac{2 - \sqrt{2}i}{3 + \sqrt{6}i}$
34. $\dfrac{3 + i}{(2 + i)^2}$
35. $\dfrac{(1 + i)^2}{(-3 + 2i)^2}$

Applications and Problem Solving

36. Electricity *Impedance* is a measure of how much hindrance there is to the flow of charge in a circuit with alternating current. The impedance Z depends on the resistance R, the reactance due to capacitance X_C, and the reactance due to inductance X_L in the circuit. The impedance is written as the complex number $Z = R + (X_L - X_C)\boldsymbol{j}$. (Electrical engineers use $\boldsymbol{j}$ to denote the imaginary unit.) In the first part of a particular series circuit, the resistance is 10 ohms, the reactance due to capacitance is 2 ohms, and the reactance due to inductance is 1 ohm. In the second part of the circuit, the respective values are 3 ohms, 1 ohm, and 1 ohm.



a. Write complex numbers that represent the impedances in the two parts of the circuit.

b. Add your answers from part a to find the total impedance in the circuit.

c. The *admittance* of an AC circuit is a measure of how well the circuit allows current to flow. Admittance is the reciprocal of impedance. That is, $S = \frac{1}{Z}$. The units for admittance are siemens. Find the admittance in a circuit with an impedance of $6 + 3\boldsymbol{j}$ ohms.

37. Critical Thinking

a. Solve the equation $x^2 + 8\boldsymbol{i}x - 25 = 0$.

b. Are the solutions complex conjugates?

c. How does your result in part b compare with what you already know about complex solutions to quadratic equations?

d. Check your solutions.

38. Critical Thinking Sometimes it is useful to separate a complex function into its real and imaginary parts. Substitute $z = x + y\boldsymbol{i}$ into the function $f(z) = z^2$ to write the equation of the function in terms of x and y only. Simplify your answer.

39. Dynamical Systems Find the first five iterates for the given function and initial value.

a. $f(z) = \boldsymbol{i}z$, $z_0 = 2 - \boldsymbol{i}$

b. $f(z) = (0.5 - 0.866\boldsymbol{i})z$, $z_0 = 1 + 0\boldsymbol{i}$

40. Critical Thinking Simplify $(1 + 2\boldsymbol{i})^{-3}$.

41. Physics One way to derive the equation of motion in a spring-mass system is to solve a *differential equation*. The solutions of such a differential equation typically involve expressions of the form $\cos \beta t + \boldsymbol{i} \sin \beta t$. You generally expect solutions that are real numbers in such a situation, so you must use algebra to eliminate the imaginary numbers. Find a relationship between the constants c_1 and c_2 such that $c_1(\cos 2t + \boldsymbol{i} \sin 2t) + c_2(\cos 2t - \boldsymbol{i} \sin 2t)$ is a real number for all values of t.

Mixed Review

42. Write the equation $6x - 2y = -3$ in polar form. *(Lesson 9-4)*

43. Graph the polar equation $r = 4\theta$. *(Lesson 9-2)*

44. Write a vector equation of the line that passes through $P(-3, 6)$ and is parallel to $\vec{\mathbf{v}}\langle 1, -4\rangle$. *(Lesson 8-6)*

45. Find an ordered triple to represent $\vec{u}$ if $\vec{u} = \frac{1}{4}\vec{v} - 2\vec{w}$, $\vec{v} = \langle -8, 6, 4 \rangle$, and $\vec{w} = \langle 2, -6, 3 \rangle$. *(Lesson 8-3)*

46. If α and β are measures of two first quadrant angles, find $\cos(\alpha + \beta)$ if $\tan \alpha = \frac{4}{3}$ and $\cot \beta = \frac{5}{12}$. *(Lesson 7-3)*

47. A twig floats on the water, bobbing up and down. The distance between its highest and lowest points is 7 centimeters. It moves from its highest point down to its lowest point and back up to its highest point every 12 seconds. Write a cosine function that models the movement of the twig in relationship to the equilibrium point. *(Lesson 6-6)*

48. **Surveying** A surveyor finds that the angle of elevation from a certain point to the top of a cliff is 60°. From a point 45 feet farther away, the angle of elevation to the top of the cliff is 52°. How high is the cliff to the nearest foot? *(Lesson 5-4)*

49. What type of polynomial function would be the best model for the set of data? *(Lesson 4-8)*

x	−3	−1	1	3	5	7	9	11
f(x)	−4	−2	3	8	6	1	−3	−8

50. **Construction** A community wants to build a second pool at their community park. Their original pool has a width 5 times its depth and a length 10 times its depth. They wish to make the second pool larger by increasing the width of the original pool by 4 feet, increasing the length by 6 feet, and increasing the depth by 2 feet. The volume of the new pool will be 3420 cubic feet. Find the dimensions of the original pool. *(Lesson 4-4)*

51. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$. *(Lesson 3-8)*

52. If $f(x) = 7 - x^2$, find $f^{-1}(x)$. *(Lesson 3-4)*

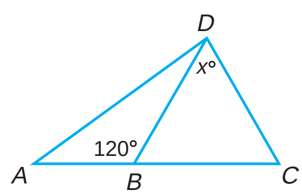
53. Find the maximum and minimum values of the function $f(x, y) = -2x + y$ for the polygonal convex set determined by the system of inequalities. *(Lesson 2-6)*

$x \leq 6$
$y \geq 1$
$y - x \leq 2$

54. Solve the system of equations. *(Lesson 2-2)*

$x + 2y - 7z = 14$
$-x - 3y + 5z = -21$
$5x - y + 2z = -7$

55. **SAT/ACT Practice** If $BC = BD$ in the figure, what is the value of $x + 40$?



A 100 **B** 80 **C** 60 **D** 40

E cannot be determined from the information given

Extra Practice See p. A43.

9-6

The Complex Plane and Polar Form of Complex Numbers

OBJECTIVES
- Graph complex numbers in the complex plane.
- Convert complex numbers from rectangular to polar form and vice versa.

Real World Application

FRACTALS One of the standard ways to generate a fractal involves iteration of a quadratic function. If the function $f(z) = z^2$ is iterated using a complex number as the initial input, there are three possible outcomes. The terms of the sequence of outputs, called the *orbit,* may

- increase in absolute value,
- decrease toward 0 in absolute value, or
- always have an absolute value of 1.

One way to analyze the behavior of the orbit is to graph the numbers in the complex plane. Plot the first five members of the orbit of $z_0 = 0.9 + 0.3\boldsymbol{i}$ under iteration by $f(z) = z^2$. *This problem will be solved in Example 3.*

Recall that $a + b\boldsymbol{i}$ is referred to as the rectangular form of a complex number. The rectangular form is sometimes written as an ordered pair, (a, b). Two complex numbers in rectangular form are equal if and only if their real parts are equal and their imaginary parts are equal.

Example 1 **Solve the equation $2x + y + 3\boldsymbol{i} = 9 + x\boldsymbol{i} - y\boldsymbol{i}$ for x and y, where x and y are real numbers.**

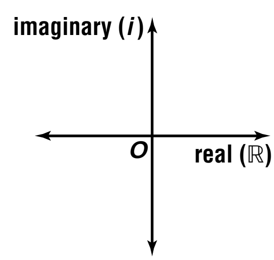
$2x + y + 3\boldsymbol{i} = 9 + x\boldsymbol{i} - y\boldsymbol{i}$

$(2x + y) + 3\boldsymbol{i} = 9 + (x - y)\boldsymbol{i}$ *On each side of the equation, group the real parts and the imaginary parts.*

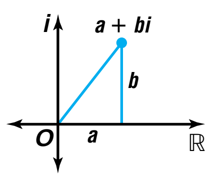
$2x + y = 9$ and $x - y = 3$ *Set the corresponding parts equal to each other.*

$x = 4$ and $y = 1$ *Solve the system of equations.*

Complex numbers can be graphed in the **complex plane.** The complex plane has a real axis and an imaginary axis. The real axis is horizontal, and the imaginary axis is vertical. The complex number $a + b\boldsymbol{i}$ is graphed as the ordered pair (a, b) in the complex plane. The complex plane is sometimes called the **Argand plane.**



Recall that the absolute value of a real number is its distance from zero on the number line. Similarly, the **absolute value** of a complex number is its distance from zero in the complex plane. When $a + b\boldsymbol{i}$ is graphed in the complex plane, the distance from zero can be calculated using the Pythagorean Theorem.

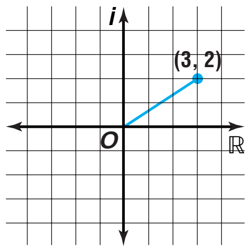


Absolute Value of a Complex Number	If $z = a + b\boldsymbol{i}$, then $\lvert z \rvert = \sqrt{a^2 + b^2}$.

Examples

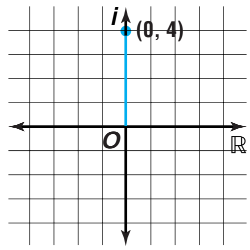
2 **Graph each number in the complex plane and find its absolute value.**

a. $\boldsymbol{z = 3 + 2i}$



$$z = 3 + 2\boldsymbol{i}$$
$$\lvert z \rvert = \sqrt{3^2 + 2^2}$$
$$= \sqrt{13}$$

b. $\boldsymbol{z = 4i}$



$$z = 0 + 4\boldsymbol{i}$$
$$\lvert z \rvert = \sqrt{0^2 + 4^2}$$
$$= 4$$

Real World Application

3 **FRACTALS** **Refer to the application at the beginning of the lesson. Plot the first five members of the orbit of $\boldsymbol{z_0 = 0.9 + 0.3i}$ under iteration by $\boldsymbol{f(z) = z^2}$.**

First, calculate the first five members of the orbit. Round the real and imaginary parts to the nearest hundredth.

$z_1 = 0.72 + 0.54\boldsymbol{i}$ $z_1 = f(z_0)$

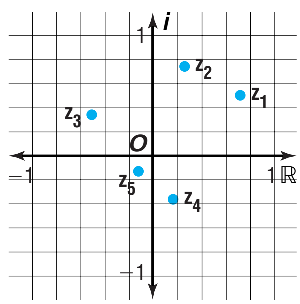
$z_2 = 0.23 + 0.78\boldsymbol{i}$ $z_2 = f(z_1)$

$z_3 = -0.55 + 0.35\boldsymbol{i}$ $z_3 = f(z_2)$

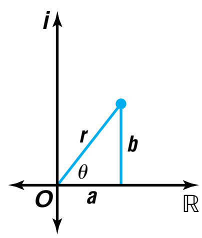
$z_4 = 0.18 - 0.39\boldsymbol{i}$ $z_4 = f(z_3)$

$z_5 = -0.12 - 0.14\boldsymbol{i}$ $z_5 = f(z_4)$

Then graph the numbers in the complex plane. The iterates approach the origin, so their absolute values decrease toward 0.



So far we have associated the complex number $a + b\boldsymbol{i}$ with the rectangular coordinates (a, b). You know from Lesson 9-1 that there are also polar coordinates (r, θ) associated with the same point. In the case of a complex number, r represents the absolute value, or **modulus,** of the complex number. The angle θ is called the **amplitude** or **argument** of the complex number. Since θ is not unique, it may be replaced by $\theta + 2\pi k$, where k is any integer.



As with other rectangular coordinates, complex coordinates can be written in polar form by substituting $a = r\cos\theta$ and $b = r\sin\theta$.

$$\begin{aligned} z &= a + b\boldsymbol{i} \\ &= r\cos\theta + (r\sin\theta)\boldsymbol{i} \\ &= r(\cos\theta + \boldsymbol{i}\sin\theta) \end{aligned}$$

This form of a complex number is often called the **polar** or **trigonometric form.**

Polar Form of a Complex Number

The polar form or trigonometric form of the complex number $a + b\boldsymbol{i}$ is

$$r(\cos\theta + \boldsymbol{i}\sin\theta).$$

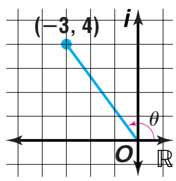
$r(\cos\theta + \boldsymbol{i}\sin\theta)$ is often abbreviated as $r \text{ cis } \theta$.

Values for r and θ can be found by using the same process you used when changing rectangular coordinates to polar coordinates. For $a + b\boldsymbol{i}$, $r = \sqrt{a^2 + b^2}$ and $\theta = \text{Arctan } \frac{b}{a}$ if $a > 0$ or $\theta = \text{Arctan } \frac{b}{a} + \pi$ if $a < 0$. The amplitude θ is usually expressed in radian measure, and the angle is in standard position along the polar axis.

Example 4 **Express each complex number in polar form.**

a. $-3 + 4\boldsymbol{i}$

First, plot the number in the complex plane.



Then find the modulus.

$r = \sqrt{(-3)^2 + 4^2}$ or 5

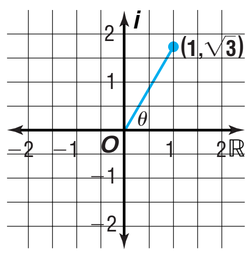
Now find the amplitude. Notice that θ is in Quadrant II.

$$\begin{aligned} \theta &= \text{Arctan } \frac{4}{-3} + \pi \\ &\approx 2.21 \end{aligned}$$

Therefore, $-3 + 4\boldsymbol{i} \approx 5(\cos 2.21 + \boldsymbol{i}\sin 2.21)$ or $5 \text{ cis } 2.21$.

b. $1 + \sqrt{3}\boldsymbol{i}$

First, plot the number in the complex plane.



Then find the modulus.

$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2}$ or 2

Now find the amplitude. Notice that θ is in Quadrant I.

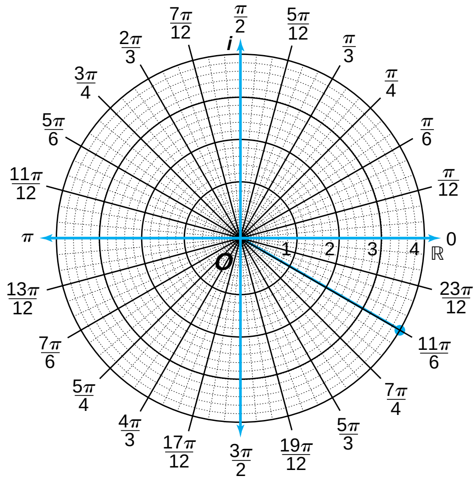
$\theta = \text{Arctan } \frac{\sqrt{3}}{1}$ or $\frac{\pi}{3}$

Therefore, $1 + \sqrt{3}\boldsymbol{i} = 2\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ or $2 \text{ cis } \frac{\pi}{3}$.

You can also graph complex numbers in polar form.

Example 5 **Graph $4\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$. Then express it in rectangular form.**

In the polar form of this complex number, the value of r is 4, and the value of θ is $\frac{11\pi}{6}$. Plot the point with polar coordinates $\left(4, \frac{11\pi}{6}\right)$.



To express the number in rectangular form, simplify the trigonometric values:

$$4\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$
$$= 4\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$
$$= 2\sqrt{3} - 2i$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to find the absolute value of a complex number.
2. **Write** the polar form of i.
3. **Find a counterexample** to the statement $|z_1 + z_2| = |z_1| + |z_2|$ *for all complex numbers* z_1 *and* z_2.
4. *Math Journal* Your friend is studying complex numbers at another school at the same time that you are. She learned that the absolute value of a complex number is the square root of the product of the number and its conjugate. You know that this is not how you learned it. Write a letter to your friend explaining why this method gives the same answer as the method you know. Use algebra, but also include some numerical examples of both techniques.

Guided Practice

5. Solve the equation $2x + y + xi + yi = 5 + 4i$ for x and y, where x and y are real numbers.

Graph each number in the complex plane and find its absolute value.

6. $-2 - i$
7. $1 + \sqrt{2}i$

Express each complex number in polar form.

8. $2 - 2i$
9. $4 + 5i$
10. -2

Graph each complex number. Then express it in rectangular form.

11. $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
12. $2(\cos 3 + i \sin 3)$
13. $\frac{3}{2}(\cos 2\pi + i \sin 2\pi)$

14. Graph the first five members of the orbit of $z_0 = -0.25 + 0.75i$ under iteration by $f(z) = z^2 + 0.5$.

15. **Vectors** The force on an object is represented by the complex number $10 + 15\boldsymbol{i}$, where the components are measured in newtons.
   a. What is the magnitude of the force?
   b. What is the direction of the force?

EXERCISES

Practice

Solve each equation for *x* and *y*, where *x* and *y* are real numbers.

16. $2x - 5y\boldsymbol{i} = 12 + 15\boldsymbol{i}$
17. $1 + (x + y)\boldsymbol{i} = y + 3x\boldsymbol{i}$
18. $4x + y\boldsymbol{i} - 5\boldsymbol{i} = 2x - y + x\boldsymbol{i} + 7\boldsymbol{i}$

Graph each number in the complex plane and find its absolute value.

19. $2 + 3\boldsymbol{i}$
20. $3 - 4\boldsymbol{i}$
21. $-1 - 5\boldsymbol{i}$
22. $-3\boldsymbol{i}$
23. $-1 + \sqrt{5}\boldsymbol{i}$
24. $4 + \sqrt{2}i$
25. Find the modulus of $z = -4 + 6\boldsymbol{i}$.

Express each complex number in polar form.

26. $3 + 3\boldsymbol{i}$
27. $-1 - \sqrt{3}\boldsymbol{i}$
28. $6 - 8\boldsymbol{i}$
29. $-4 + \boldsymbol{i}$
30. $20 - 21\boldsymbol{i}$
31. $-2 + 4\boldsymbol{i}$
32. 3
33. $-4\sqrt{2}$
34. $-2\boldsymbol{i}$

Graph each complex number. Then express it in rectangular form.

35. $3\left(\cos \frac{\pi}{4} + \boldsymbol{i} \sin \frac{\pi}{4}\right)$
36. $\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)$
37. $2\left(\cos \frac{4\pi}{3} + \boldsymbol{i} \sin \frac{4\pi}{3}\right)$
38. $10(\cos 6 + \boldsymbol{i} \sin 6)$
39. $2\left(\cos \frac{5\pi}{4} + \boldsymbol{i} \sin \frac{5\pi}{4}\right)$
40. $2.5(\cos 1 + \boldsymbol{i} \sin 1)$
41. $5(\cos 0 + \boldsymbol{i} \sin 0)$
42. $3(\cos \pi + \boldsymbol{i} \sin \pi)$

Graph the first five members of the orbit of each initial value under iteration by the given function.

43. $z_0 = -0.5 + \boldsymbol{i}, f(z) = z^2 + 0.5$
44. $z_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}, f(z) = z^2$
45. Graph the first five iterates of $z_0 = 0.5 - 0.5\boldsymbol{i}$ under $f(z) = z^2 - 0.5$.

Applications and Problem Solving

46. **Electrical Engineering** Refer to Exercise 44 in Lesson 9-3. Consider a circuit with alternating current that contains two voltage sources in series. Suppose these two voltages are given by $v_1(t) = 40 \sin (250t + 30°)$ and $v_2(t) = 60 \sin (250t + 60°)$, where t represents time, in seconds.
   a. The phasors for these two voltage sources are written as $40\angle 30°$ and $60\angle 60°$, respectively. Convert these phasors to complex numbers in rectangular form. (Use $\boldsymbol{j}$ as the imaginary unit, as electrical engineers do.)
   b. Add these two complex numbers to find the total voltage in the circuit.
   c. Write a sinusoidal function that gives the total voltage in the circuit.

47. Critical Thinking How are the polar forms of complex conjugates alike? How are they different?

48. Electricity A series circuit contains two sources of impedance, one of $10(\cos 0.7 + \boldsymbol{j} \sin 0.7)$ ohms and the other of $16(\cos 0.5 + \boldsymbol{j} \sin 0.5)$ ohms.

a. Convert these complex numbers to rectangular form.

b. Add your answers from part a to find the total impedance in the circuit.

c. Convert the total impedance back to polar form.

49. Transformations Certain operations with complex numbers correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (2 - 3\boldsymbol{i})$

b. $w = \boldsymbol{i} \cdot z$

c. $w = 3z$

d. w is the conjugate of z

50. Critical Thinking Choose any two complex numbers, z_1 and z_2, in rectangular form.

a. Find the product z_1z_2.

b. Write z_1, z_2, and z_1z_2 in polar form.

c. Repeat this procedure with a different pair of complex numbers.

d. Make a conjecture about the product of two complex numbers in polar form.

Mixed Review

51. Simplify $(6 - 2\boldsymbol{i})(-2 + 3\boldsymbol{i})$. *(Lesson 9-5)*

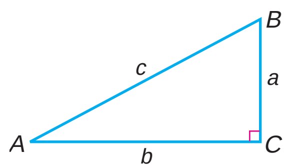
52. Find the rectangular coordinates of the point with polar coordinates $(-3, -135°)$. *(Lesson 9-3)*

53. Find the magnitude of the vector $\langle -3, 7 \rangle$, and write the vector as a sum of unit vectors. *(Lesson 8-2)*

54. Use a sum or difference identity to find tan 105°. *(Lesson 7-3)*

55. Mechanics A pulley of radius 18 centimeters turns at 12 revolutions per second. What is the linear velocity of the belt driving the pulley in meters per second? *(Lesson 6-2)*

56. If $a = 12$ and $c = 18$ in $\triangle ABC$, find the measure of angle A to the nearest tenth of a degree. *(Lesson 5-5)*



57. Solve $\sqrt{2a - 1} = \sqrt{3a - 5}$. *(Lesson 4-7)*

58. Without graphing, describe the end behavior of the graph of $y = 2x^2 + 2$. *(Lesson 3-5)*

59. SAT/ACT Practice A person is hired for a job that pays $500 per month and receives a 10% raise in each following month. In the fourth month, how much will that person earn?

A $550 **B** $600.50 **C** $650.50 **D** $665.50 **E** $700

Extra Practice See p. A43.

GRAPHING CALCULATOR EXPLORATION

9-6B Geometry in the Complex Plane

OBJECTIVE

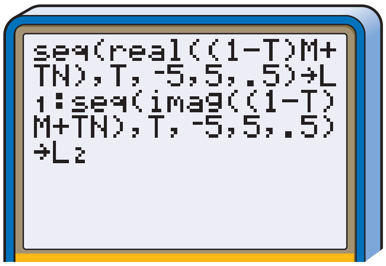
- Explore geometric relationships in the complex plane.

An Extension of Lesson 9-6

Many geometric figures and relationships can be described by using complex numbers. To show points on figures, you can store the real and imaginary parts of the complex numbers that correspond to the points in lists **L1** and **L2** and use **STAT PLOT** to graph the points.

TRY THESE

1. Store $-1 + 2\boldsymbol{i}$ as M and $1 + 5\boldsymbol{i}$ as N. Now consider complex numbers of the form $(1 - T)M + TN$, where T is a real number. You can generate several numbers of this form and store their real and imaginary parts in **L1** and **L2**, respectively, by entering the following instructions on the home screen.



__seq(__ is in the __LIST OPS__ menu.
__real(__ and __imag(__ are in the __MATH CPX__ menu.

Use a graphing window of $[-10, 10]$ scl:1 by $[-25, 25]$ scl:5. Turn on **Plot 1** and use a scatter plot to display the points defined in **L1** and **L2**. What do you notice about the points in the scatter plot?

2. Are the original numbers M and N shown in the scatter plot? Explain.
3. Repeat Exercise 1 storing $-1 + 1.5\boldsymbol{i}$ as M and $-2 - \boldsymbol{i}$ as N. Describe your results.
4. Repeat Exercises 1 and 2 for several complex numbers M and N of your choice. (You may need to change the window settings.) Then make a conjecture about where points of the form $(1 - T)M + TN$ are located in relation to M and N.
5. Suppose K, M, and N are three noncollinear points in the complex plane. Where will you find all the points that can be expressed in the form $aK + bM + cN$, where a, b, and c are nonnegative real numbers such that $a + b + c = 1$? Use the calculator to check your answer.

WHAT DO YOU THINK?

6. In Exercises 1-4, where is $(1 - T)M + TN$ in relation to M and N if the value of T is between 0 and 1?
7. Where in the complex plane will you find the complex numbers z that satisfy the equation $|z - (1 - \boldsymbol{i})| = 5$?
8. What equation models the points in the complex plane that lie on the circle of radius 2 that is centered at the point $2 + 3\boldsymbol{i}$?

9-7 Products and Quotients of Complex Numbers in Polar Form

OBJECTIVE

- Find the product and quotient of complex numbers in polar form.

Real World Application

ELECTRICITY Complex numbers can be used in the study of electricity, specifically alternating current (AC). There are three basic quantities to consider:

- the *current I*, measured in amperes,
- the *impedance Z* to the current, measured in ohms, and
- the *electromotive force E* or *voltage*, measured in volts.

These three quantities are related by the equation $E = I \cdot Z$. Current, impedance, and voltage can be expressed as complex numbers. Electrical engineers use $\boldsymbol{j}$ as the imaginary unit, so they write complex numbers in the form $a + bj$. For the total impedance $a + bj$, the real part a represents the opposition to current flow due to resistors, and the imaginary part b is related to the opposition due to inductors and capacitors. If a circuit has a total impedance of $2 - 6\boldsymbol{j}$ ohms and a voltage of 120 volts, find the current in the circuit. *This problem will be solved in Example 3.*

Multiplication and division of complex numbers in polar form are closely tied to geometric transformations in the complex plane. Let $r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)$ and $r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)$ be two complex numbers in polar form. A formula for the product of the two numbers can be derived by multiplying the two numbers directly and simplifying the result.

$$
\begin{aligned}
& r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1) \cdot r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2) \\
&= r_1 r_2(\cos \theta_1 \cos \theta_2 + \boldsymbol{i} \cos \theta_1 \sin \theta_2 + \boldsymbol{i} \sin \theta_1 \cos \theta_2 + \boldsymbol{i}^2 \sin \theta_1 \sin \theta_2) \\
&= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + \boldsymbol{i}(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \quad \boldsymbol{i}^2 = -1 \\
&= r_1 r_2[\cos (\theta_1 + \theta_2) + \boldsymbol{i} \sin (\theta_1 + \theta_2)] \quad \text{Sum identities for cosine and sine}
\end{aligned}
$$

Product of Complex Numbers in Polar Form

$$r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1) \cdot r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2) = r_1 r_2[\cos (\theta_1 + \theta_2) + \boldsymbol{i} \sin (\theta_1 + \theta_2)]$$

Notice that the modulus $(r_1 r_2)$ of the product of the two complex numbers is the product of their moduli. The amplitude $(\theta_1 + \theta_2)$ of the product is the sum of the amplitudes.

Example 1 **Find the product $3\left(\cos \frac{7\pi}{6} + \boldsymbol{i} \sin \frac{7\pi}{6}\right) \cdot 2\left(\cos \frac{2\pi}{3} + \boldsymbol{i} \sin \frac{2\pi}{3}\right)$. Then express the product in rectangular form.**

Find the modulus and amplitude of the product.

$$r = r_1 r_2 = 3(2) = 6 \qquad \theta = \theta_1 + \theta_2 = \frac{7\pi}{6} + \frac{2\pi}{3} = \frac{11\pi}{6}$$

The product is $6\left(\cos \frac{11\pi}{6} + \boldsymbol{i} \sin \frac{11\pi}{6}\right)$.

Now find the rectangular form of the product.

$$6\left(\cos \frac{11\pi}{6} + \boldsymbol{i} \sin \frac{11\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right) \qquad \cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2},\ \sin \frac{11\pi}{6} = -\frac{1}{2}$$

$$= 3\sqrt{3} - 3\boldsymbol{i}$$

The rectangular form of the product is $3\sqrt{3} - 3\boldsymbol{i}$.

Suppose the quotient of two complex numbers is expressed as a fraction. A formula for this quotient can be derived by rationalizing the denominator. To rationalize the denominator, multiply both the numerator and denominator by the same value so that the resulting new denominator does not contain imaginary numbers.

$$\frac{r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)}{r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)}$$

$$= \frac{r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)}{r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)} \cdot \frac{(\cos \theta_2 - \boldsymbol{i} \sin \theta_2)}{(\cos \theta_2 - \boldsymbol{i} \sin \theta_2)} \qquad \cos \theta_2 - \boldsymbol{i} \sin \theta_2 \text{ is the conjugate of } \cos \theta_2 + \boldsymbol{i} \sin \theta_2.$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + \boldsymbol{i}(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)] \qquad \textit{Trigonometric identities}$$

Quotient of Complex Numbers in Polar Form

$$\frac{r_1[\cos \theta_1 + \boldsymbol{i} \sin \theta_1]}{r_2[\cos \theta_2 + \boldsymbol{i} \sin \theta_2]} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)]$$

Notice that the modulus $\left(\frac{r_1}{r_2}\right)$ of the quotient of two complex numbers is the quotient of their moduli. The amplitude $(\theta_1 - \theta_2)$ of the quotient is the difference of the amplitudes.

Example 2 **Find the quotient $12\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \div 4\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$. Then express the quotient in rectangular form.**

Find the modulus and amplitude of the quotient.

$$r = \frac{r_1}{r_2} = \frac{12}{4} = 3 \qquad \theta = \theta_1 - \theta_2 = \frac{\pi}{4} - \frac{3\pi}{2} = -\frac{5\pi}{4}$$

The quotient is $3\left[\cos\left(-\frac{5\pi}{4}\right) + i \sin\left(-\frac{5\pi}{4}\right)\right]$.

Now find the rectangular form of the quotient.

$$3\left[\cos\left(-\frac{5\pi}{4}\right) + i \sin\left(-\frac{5\pi}{4}\right)\right] = 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$= -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

The rectangular form of the quotient is $-\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$.

You can use products and quotients of complex numbers in polar form to solve the problem presented at the beginning of the lesson.

Example 3 Real World Application

ELECTRICITY **If a circuit has an impedance of $2 - 6j$ ohms and a voltage of 120 volts, find the current in the circuit.**

Express each complex number in polar form.

$$120 = 120(\cos 0 + j \sin 0)$$

$$2 - 6j \approx \sqrt{40}[\cos(-1.25) + j \sin(-1.25)] \qquad r = \sqrt{2^2 + (-6)^2} = \sqrt{40} \text{ or } 2\sqrt{10},$$

$$\approx 2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)] \qquad \theta = \text{Arctan}\,\frac{-6}{2} \text{ or } -1.25$$

Substitute the voltage and impedance into the equation $E = I \cdot Z$.

$$E = I \cdot Z$$

$$120(\cos 0 + j \sin 0) = I \cdot 2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)]$$

$$\frac{120(\cos 0 + j \sin 0)}{2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)]} = I$$

$$6\sqrt{10}(\cos 1.25 + j \sin 1.25) = I$$

Now express the current in rectangular form.

$$I = 6\sqrt{10}(\cos 1.25 + j \sin 1.25)$$

$$\approx 5.98 + 18.01j$$

Use a calculator. $6\sqrt{10} \cos 1.25 \approx 5.98$, $6\sqrt{10} \sin 1.25 \approx 18.01$

The current is about $6 + 18j$ amps.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to find the quotient of two complex numbers in polar form.
2. **Describe** how to square a complex number in polar form.
3. **List** which operations with complex numbers you think are easier in rectangular form and which you think are easier in polar form. Defend your choices with examples.

Guided Practice

Find each product or quotient. Express the result in rectangular form.

4. $2\left(\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}\right) \cdot 2\left(\cos \frac{3\pi}{2} + \boldsymbol{i} \sin \frac{3\pi}{2}\right)$

5. $3\left(\cos \frac{\pi}{6} + \boldsymbol{i} \sin \frac{\pi}{6}\right) \div 4\left(\cos \frac{2\pi}{3} + \boldsymbol{i} \sin \frac{2\pi}{3}\right)$

6. $4\left(\cos \frac{9\pi}{4} + \boldsymbol{i} \sin \frac{9\pi}{4}\right) \div 2\left[\cos\left(-\frac{\pi}{2}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{2}\right)\right]$

7. $\frac{1}{2}\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right) \cdot 6\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right)$

8. Use polar form to find the product $\left(2 + 2\sqrt{3}\boldsymbol{i}\right) \cdot \left(-3 + \sqrt{3}\boldsymbol{i}\right)$. Express the result in rectangular form.

9. **Electricity** Determine the voltage in a circuit when there is a current of $2\left(\cos \frac{11\pi}{6} + \boldsymbol{j} \sin \frac{11\pi}{6}\right)$ amps and an impedance of $3\left(\cos \frac{\pi}{3} + \boldsymbol{j} \sin \frac{\pi}{3}\right)$ ohms.

EXERCISES

Practice

Find each product or quotient. Express the result in rectangular form.

10. $4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right) \cdot 7\left(\cos \frac{2\pi}{3} + \boldsymbol{i} \sin \frac{2\pi}{3}\right)$

11. $6\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right) \div 2\left(\cos \frac{\pi}{4} + \boldsymbol{i} \sin \frac{\pi}{4}\right)$

12. $\frac{1}{2}\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right) \div 3\left(\cos \frac{\pi}{6} + \boldsymbol{i} \sin \frac{\pi}{6}\right)$

13. $5(\cos \pi + \boldsymbol{i} \sin \pi) \cdot 2\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right)$

14. $6\left[\cos\left(-\frac{\pi}{3}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{3}\right)\right] \cdot 3\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right)$

15. $3\left(\cos \frac{7\pi}{3} + \boldsymbol{i} \sin \frac{7\pi}{3}\right) \div \left(\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}\right)$

16. $2(\cos 240^\circ + \boldsymbol{i} \sin 240^\circ) \cdot 3(\cos 60^\circ + \boldsymbol{i} \sin 60^\circ)$

17. $\sqrt{2}\left(\cos \frac{7\pi}{4} + \boldsymbol{i} \sin \frac{7\pi}{4}\right) \div \frac{\sqrt{2}}{2}\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right)$

18. $3(\cos 4 + \boldsymbol{i} \sin 4) \cdot 0.5(\cos 2.5 + \boldsymbol{i} \sin 2.5)$

www.amc.glencoe.com/self_check_quiz

CONTENTS

19. $4[\cos(-2) + \boldsymbol{i}\sin(-2)] \div (\cos 3.6 + \boldsymbol{i}\sin 3.6)$

20. $20\left(\cos\frac{7\pi}{6} + \boldsymbol{i}\sin\frac{7\pi}{6}\right) \div 15\left(\cos\frac{11\pi}{3} + \boldsymbol{i}\sin\frac{11\pi}{3}\right)$

21. $2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right) \cdot \sqrt{2}\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$

22. Find the product of $2\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ and $6\left[\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{6}\right)\right]$. Write the answer in rectangular form.

23. If $z_1 = 4\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right)$ and $z_2 = \frac{1}{2}\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$, find $\frac{z_1}{z_2}$ and express the result in rectangular form.

Use polar form to find each product or quotient. Express the result in rectangular form.

24. $(2 - 2\boldsymbol{i}) \cdot (-3 + 3\boldsymbol{i})$

25. $\left(\sqrt{2} - \sqrt{2}\boldsymbol{i}\right) \cdot \left(-3\sqrt{2} - 3\sqrt{2}\boldsymbol{i}\right)$

26. $\left(\sqrt{3} - \boldsymbol{i}\right) \div \left(2 - 2\sqrt{3}\boldsymbol{i}\right)$

27. $\left(-4\sqrt{2} + 4\sqrt{2}\boldsymbol{i}\right) \div (6 + 6\boldsymbol{i})$

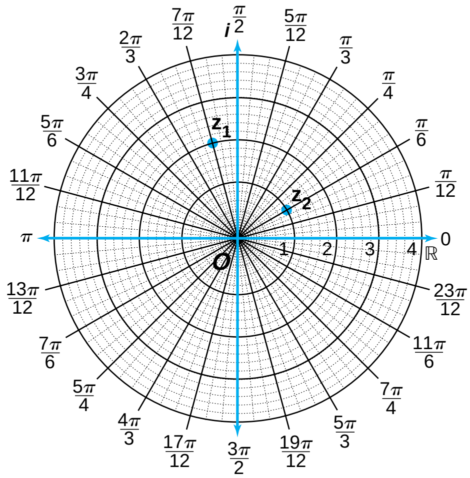
Applications and Problem Solving

Real World Application

28. **Electricity** Find the current in a circuit with a voltage of 13 volts and an impedance of $3 - 2\boldsymbol{j}$ ohms.

29. **Electricity** Find the impedance in a circuit with a voltage of 100 volts and a current of $4 - 3\boldsymbol{j}$ amps.

30. **Critical Thinking** Given z_1 and z_2 graphed at the right, graph z_1z_2 and $\frac{z_1}{z_2}$ without actually calculating them.



31. **Transformations**
    a. Describe the transformation applied to the graph of the complex number z if z is multiplied by $\cos\theta + \boldsymbol{i}\sin\theta$.
    b. Describe the transformation applied to the graph of the complex number z if z is multiplied by $\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}$.

32. **Critical Thinking** Find the quadratic equation $az^2 + bz + c = 0$ such that $a = 1$ and the solutions are $3\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ and $2\left(\cos\frac{5\pi}{6} + \boldsymbol{i}\sin\frac{5\pi}{6}\right)$.

Mixed Review

33. Express $5 - 12\boldsymbol{i}$ in polar form. *(Lesson 9-6)*

34. Write the equation $r = 5\sec\left(\theta - \frac{5\pi}{6}\right)$ in rectangular form. *(Lesson 9-4)*

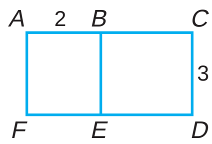
35. **Physics** A prop for a play is supported equally by two wires suspended from the ceiling. The wires form a 130° angle with each other. If the prop weighs 23 pounds, what is the tension in each of the wires? *(Lesson 8-5)*

Extra Practice See p. A43.

36. Solve cos $2x + \sin x = 1$ for principal values of x. *(Lesson 7-5)*

37. Write the equation for the inverse of $y = \cos x$. *(Lesson 6-8)*

38. **SAT/ACT Practice** In the figure, the perimeter of square *BCDE* is how much smaller than the perimeter of rectangle *ACDF*?



A 2 **B** 3 **C** 4

D 6 **E** 16

CAREER CHOICES

Astronomer

Have you ever gazed into the sky at night hoping to spot a constellation? Do you dream of having your own telescope? If you enjoy studying about the universe, then a career in astronomy may be just for you. Astronomers collect and analyze data about the universe including stars, planets, comets, asteroids, and even artificial satellites. As an astronomer, you may collect information by using a telescope or spectrometer here on earth, or you may use information collected by spacecraft and satellites.

Most astronomers specialize in one branch of astronomy such as astrophysics or celestial mechanics. Astronomers often teach in addition to conducting research. Astronomers located throughout the world are prime sources of information for NASA and other countries' space programs.

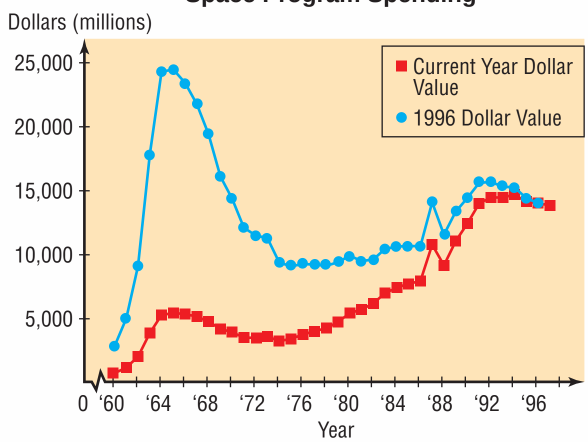
Career Overview

Degree Preferred:
at least a bachelor's degree in astronomy or physics

Related Courses:
mathematics, physics, chemistry, computer science

Outlook:
average through the year 2006

Space Program Spending



Source: National Aeronautics and Space Administration

*inter*NET CONNECTION For more information on careers in astronomy, visit: **www.amc.glencoe.com**

9-8 Powers and Roots of Complex Numbers

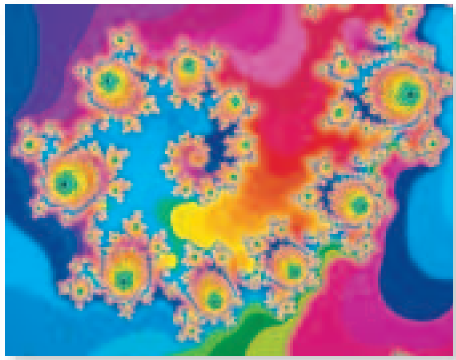
OBJECTIVE

- Find powers and roots of complex numbers in polar form using De Moivre's Theorem.

Real World Application

COMPUTER GRAPHICS Many of the computer graphics that are referred to as fractals are graphs of Julia sets, which are named after the mathematician Gaston Julia. When a function like $f(z) = z^2 + c$, where c is a complex constant, is iterated, points in the complex plane can be classified according to their behavior under iteration.

- Points that escape to infinity under iteration belong to the **escape set** of the function.
- Points that do not escape belong to the **prisoner set.**



The **Julia set** is the boundary between the escape set and the prisoner set. Is the number $w = 0.6 - 0.5\boldsymbol{i}$ in the escape set or the prisoner set of the function $f(z) = z^2$? *This problem will be solved in Example 6.*

You can use the formula for the product of complex numbers to find the square of a complex number.

$$\begin{aligned}[r(\cos\theta + \boldsymbol{i}\sin\theta)]^2 &= [r(\cos\theta + \boldsymbol{i}\sin\theta)]\cdot[r(\cos\theta + \boldsymbol{i}\sin\theta)]\\ &= r^2[\cos(\theta+\theta) + \boldsymbol{i}\sin(\theta+\theta)]\\ &= r^2(\cos 2\theta + \boldsymbol{i}\sin 2\theta)\end{aligned}$$

Other powers of complex numbers can be found using De Moivre's Theorem.

De Moivre's Theorem

$$[r(\cos\theta + \boldsymbol{i}\sin\theta)]^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$$

You will be asked to prove De Moivre's Theorem in Chapter 12.

Example 1 **Find $\left(2 + 2\sqrt{3}\boldsymbol{i}\right)^6$.**

First, write $2 + 2\sqrt{3}\boldsymbol{i}$ in polar form. Note that its graph is in the first quadrant of the complex plane.

$$\begin{aligned} r &= \sqrt{2^2 + \left(2\sqrt{3}\right)^2} & \theta &= \operatorname{Arctan}\frac{2\sqrt{3}}{2}\\ &= \sqrt{4+12} & &= \operatorname{Arctan}\sqrt{3}\\ &= 4 & &= \frac{\pi}{3}\end{aligned}$$

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The polar form of $2 + 2\sqrt{3}\boldsymbol{i}$ is $4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)$.

Now use De Moivre's Theorem to find the sixth power.

$$\begin{aligned}(2 + 2\sqrt{3}\boldsymbol{i})^6 &= \left[4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)\right]^6 \\ &= 4^6\left[\cos 6\left(\frac{\pi}{3}\right) + \boldsymbol{i} \sin 6\left(\frac{\pi}{3}\right)\right] \\ &= 4096(\cos 2\pi + \boldsymbol{i} \sin 2\pi) \\ &= 4096(1 + 0\boldsymbol{i}) \quad \textit{Write the result in rectangular form.} \\ &= 4096\end{aligned}$$

Therefore, $\left(2 + 2\sqrt{3}\boldsymbol{i}\right)^6 = 4096$.

De Moivre's Theorem is valid for all rational values of n. Therefore, it is also useful for finding negative powers of complex numbers and roots of complex numbers.

Example 2 **Find** $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right)^{-5}$.

First, write $\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}$ in polar form. Note that its graph is in the fourth quadrant of the complex plane.

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \qquad \theta = \text{Arctan}\ \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} \text{ or } 1 \qquad = \text{Arctan}\left(-\frac{\sqrt{3}}{3}\right) \text{ or } -\frac{\pi}{6}$$

The polar form of $\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}$ is $1\left[\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)\right]$.

Use De Moivre's Theorem to find the negative 5th power.

$$\begin{aligned}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right)^{-5} &= \left[1\left(\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)\right)\right]^{-5} \\ &= 1^{-5}\left[\cos (-5)\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin (-5)\left(-\frac{\pi}{6}\right)\right] \quad \textit{De Moivre's Theorem} \\ &= 1\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right) \quad \textit{Simplify.} \\ &= -\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i} \quad \textit{Write the answer in rectangular form.}\end{aligned}$$

Recall that positive real numbers have two square roots and that the positive one is called the principal square root. In general, all nonzero complex numbers have p distinct pth roots. That is, they each have two square roots, three cube roots, four fourth roots, and so on. The principal pth root of a complex number is given by:

$$(a + b\boldsymbol{i})^{\frac{1}{p}} = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^{\frac{1}{p}}$$
$$= r^{\frac{1}{p}}\left(\cos\frac{\theta}{p} + \boldsymbol{i}\sin\frac{\theta}{p}\right).$$

When finding a principal root, the interval $-\pi < \theta \le \pi$ is used.

Example 3 **Find $\sqrt[3]{8\boldsymbol{i}}$.**

$\sqrt[3]{8\boldsymbol{i}} = (0 + 8\boldsymbol{i})^{\frac{1}{3}}$ — *$a = 0$, $b = 8$*

$= \left[8\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$ — *Polar form; $r = \sqrt{0^2 + 8^2}$ or 8, $\theta = \frac{\pi}{2}$ since $a = 0$.*

$= 8^{\frac{1}{3}}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) + \boldsymbol{i}\sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\right]$ — *De Moivre's Theorem*

$= 2\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}\right)$ or $\sqrt{3} + \boldsymbol{i}$ — *This is the principal cube root.*

The following formula generates all of the pth roots of a complex number. It is based on the identities $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, where n is any integer.

The p Distinct pth Roots of a Complex Number

The p distinct pth roots of $a + b\boldsymbol{i}$ can be found by replacing n with 0, 1, 2, …, $p - 1$, successively, in the following equation.

$$(a + b\boldsymbol{i})^{\frac{1}{p}} = (r[\cos(\theta + 2n\pi) + \boldsymbol{i}\sin(\theta + 2n\pi)])^{\frac{1}{p}}$$
$$= r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + \boldsymbol{i}\sin\frac{\theta + 2n\pi}{p}\right)$$

Example 4 **Find the three cube roots of $-2 - 2\boldsymbol{i}$.**

First, write $-2 - 2\boldsymbol{i}$ in polar form.

$r = \sqrt{(-2)^2 + (-2)^2}$ or $2\sqrt{2}$ $\qquad$ $\theta = \text{Arctan}\frac{-2}{-2} + \pi$ or $\frac{5\pi}{4}$

$-2 - 2\boldsymbol{i} = 2\sqrt{2}\left[\cos\left(\frac{5\pi}{4} + 2n\pi\right) + \boldsymbol{i}\sin\left(\frac{5\pi}{4} + 2n\pi\right)\right]$ — *n is any integer.*

Now write an expression for the cube roots.

$(-2 - 2\boldsymbol{i})^{\frac{1}{3}} = \left(2\sqrt{2}\left[\cos\left(\frac{5\pi}{4} + 2n\pi\right) + \boldsymbol{i}\sin\left(\frac{5\pi}{4} + 2n\pi\right)\right]\right)^{\frac{1}{3}}$

$= \sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2n\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2n\pi}{3}\right)\right]$ — *$\left(2\sqrt{2}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}}$ or $\sqrt{2}$*

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Let $n = 0$, 1, and 2 successively to find the cube roots.

Let $n = 0$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(0)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(0)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{5\pi}{12} + \boldsymbol{i}\sin\frac{5\pi}{12}\right)$$
$$\approx 0.37 + 1.37\boldsymbol{i}$$

Let $n = 1$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(1)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(1)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{13\pi}{12} + \boldsymbol{i}\sin\frac{13\pi}{12}\right)$$
$$\approx -1.37 - 0.37i$$

Let $n = 2$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(2)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(2)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{21\pi}{12} + \boldsymbol{i}\sin\frac{21\pi}{12}\right)$$
$$= 1 - \boldsymbol{i}$$

The cube roots of $-2 - 2\boldsymbol{i}$ are approximately $0.37 + 1.37\boldsymbol{i}$, $-1.37 - 0.37\boldsymbol{i}$, and $1 - \boldsymbol{i}$. *These roots can be checked by multiplication.*

GRAPHING CALCULATOR EXPLORATION

The p distinct pth roots of a complex number can be approximated using the parametric mode on a graphing calculator. For a particular complex number $r(\cos\theta + \boldsymbol{i}\sin\theta)$ and a particular value of p:

- Select the **Radian** and **Par** modes.
- Select the viewing window.

  $\text{Tmin} = \frac{\theta}{p}$, $\text{Tmax} = \frac{\theta}{p} + 2\pi$, $\text{Tstep} = \frac{2\pi}{p}$,

  $\text{Xmin} = -r^{\frac{1}{p}}$, $\text{Xmax} = r^{\frac{1}{p}}$, $\text{Xscl} = 1$,

  $\text{Ymin} = -r^{\frac{1}{p}}$, $\text{Ymax} = r^{\frac{1}{p}}$, and $\text{Yscl} = 1$.
- Enter the parametric equations $X_{1T} = r^{\frac{1}{p}}\cos T$ and $Y_{1T} = r^{\frac{1}{p}}\sin T$.
- Graph the equations.
- Use TRACE to locate the roots.

TRY THESE

1. Approximate the cube roots of 1.

2. Approximate the fourth roots of $\boldsymbol{i}$.

3. Approximate the fifth roots of $1 + \boldsymbol{i}$.

WHAT DO YOU THINK?

4. What geometric figure is formed when you graph the three cube roots of a complex number?

5. What geometric figure is formed when you graph the fifth roots of a complex number?

6. Under what conditions will the complex number $a + b\boldsymbol{i}$ have a root that lies on the positive real axis?

You can also use De Moivre's Theorem to solve some polynomial equations.

Examples

5 **Solve $x^5 - 32 = 0$. Then graph the roots in the complex plane.**

The solutions to this equation are the same as those of the equation $x^5 = 32$. That means we have to find the fifth roots of 32.

$32 = 32 + 0\boldsymbol{i}$ *$a = 32, b = 0$*

$= 32(\cos 0 + \boldsymbol{i} \sin 0)$ *Polar form; $r = \sqrt{32^2 + 0^2}$ or 32, $\theta = \text{Arctan } \frac{0}{32}$ or 0*

Now write an expression for the fifth roots.

$$32^{\frac{1}{5}} = [32(\cos (0 + 2n\pi) + \boldsymbol{i} \sin (0 + 2n\pi))]^{\frac{1}{5}}$$
$$= 2\left(\cos \frac{2n\pi}{5} + \boldsymbol{i} \sin \frac{2n\pi}{5}\right)$$

Let $n = 0, 1, 2, 3,$ and 4 successively to find the fifth roots, $x_1, x_2, x_3, x_4,$ and x_5.

Let $n = 0$. $x_1 = 2(\cos 0 + \boldsymbol{i} \sin 0) = 2$

Let $n = 1$. $x_2 = 2\left(\cos \frac{2\pi}{5} + \boldsymbol{i} \sin \frac{2\pi}{5}\right) \approx 0.62 + 1.90\boldsymbol{i}$

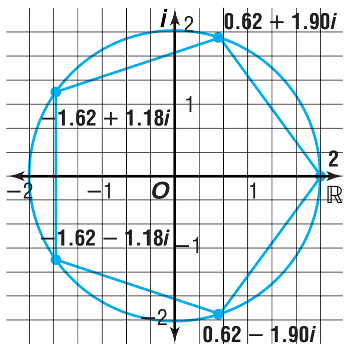
Let $n = 2$. $x_3 = 2\left(\cos \frac{4\pi}{5} + \boldsymbol{i} \sin \frac{4\pi}{5}\right) \approx -1.62 + 1.18\boldsymbol{i}$

Let $n = 3$. $x_4 = 2\left(\cos \frac{6\pi}{5} + \boldsymbol{i} \sin \frac{6\pi}{5}\right) \approx -1.62 - 1.18\boldsymbol{i}$

Let $n = 4$. $x_5 = 2\left(\cos \frac{8\pi}{5} + \boldsymbol{i} \sin \frac{8\pi}{5}\right) \approx 0.62 - 1.90\boldsymbol{i}$

The solutions of $x^5 - 32 = 0$ are 2, $0.62 \pm 1.90\boldsymbol{i}$, and $-1.62 \pm 1.18\boldsymbol{i}$.

The solutions are graphed at the right. Notice that the points are the vertices of a regular pentagon. The roots of a complex number are cyclical in nature. That means, when the roots are graphed on the complex plane, the roots are equally spaced around a circle.



6 **COMPUTER GRAPHICS** **Refer to the application at the beginning of the lesson. Is the number $w = 0.6 - 0.5i$ in the escape set or the prisoner set of the function $f(z) = z^2$?**

Iterating this function requires you to square complex numbers, so you can use De Moivre's Theorem.

Write w in polar form. *$r = \sqrt{0.6^2 + (-0.5)^2}$ or about 0.78*

$w = 0.78[\cos (-0.69) + \boldsymbol{i} \sin (-0.69)]$ *$\theta = \text{Arctan } \frac{-0.5}{0.6}$ or about -0.69*

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interNET CONNECTION

Graphing Calculator Programs
For a program that draws Julia sets, visit: **www.amc.glencoe.com**

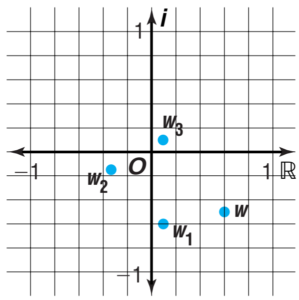
Now iterate the function.

$w_1 = f(w)$
$= w^2$
$= (0.78[\cos(-0.69) + \boldsymbol{i}\sin(-0.69)])^2$
$= 0.78^2[\cos 2(-0.69) + \boldsymbol{i}\sin 2(-0.69)]$ *De Moivre's Theorem*
$= 0.12 - 0.60\boldsymbol{i}$ *Use a calculator to approximate the rectangular form.*

$w_2 = f(w_1)$
$= {w_1}^2$
$= (0.78^2[\cos 2(-0.69) + \boldsymbol{i}\sin 2(-0.69)])^2$ *Use the polar form of w_1.*
$= 0.78^4[\cos 4(-0.69) + \boldsymbol{i}\sin 4(-0.69)]$
$= -0.34 - 0.14\boldsymbol{i}$

$w_3 = f(w_2)$
$= {w_2}^2$
$= (0.78^4[\cos 4(-0.69) + \boldsymbol{i}\sin 4(-0.69)])^2$ *Use the polar form of w_2.*
$= 0.78^8[\cos 8(-0.69) + \boldsymbol{i}\sin 8(-0.69)]$
$= 0.10 + 0.09\boldsymbol{i}$

The moduli of these iterates are 0.78^2, 0.78^4, 0.78^8, and so on. These moduli will approach 0 as the number of iterations increases. This means the graphs of the iterates approach the origin in the complex plane, so $w = 0.6 - 0.5\boldsymbol{i}$ is in the prisoner set of the function.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Evaluate** the product $(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})$ by traditional multiplication. Compare the results with the results using De Moivre's Theorem on $(1 + \boldsymbol{i})^5$. Which method do you prefer?
2. **Explain** how to use De Moivre's Theorem to find the reciprocal of a complex number in polar form.
3. **Graph** all the fourth roots of a complex number if $a + a\boldsymbol{i}$ is one of the fourth roots. Assume a is positive.
4. **You Decide** Shembala says that if $a \neq 0$, then $(a + a\boldsymbol{i})^2$ must be a pure imaginary number. Arturo disagrees. Who is correct? Use polar form to explain.

Guided Practice

Find each power. Express the result in rectangular form.

5. $\left(\sqrt{3} - \boldsymbol{i}\right)^3$

6. $(3 - 5\boldsymbol{i})^4$

Find each principal root. Express the result in the form *a* + *bi* with *a* and *b* rounded to the nearest hundredth.

7. $\boldsymbol{i}^{\frac{1}{6}}$

8. $(-2 - \boldsymbol{i})^{\frac{1}{3}}$

Solve each equation. Then graph the roots in the complex plane.

9. $x^4 + \boldsymbol{i} = 0$

10. $2x^3 + 4 + 2\boldsymbol{i} = 0$

11. Fractals Refer to the application at the beginning of the lesson. Is $w = 0.8 - 0.7\boldsymbol{i}$ in the prisoner set or the escape set for the function $f(z) = z^2$? Explain.

EXERCISES

Practice

Find each power. Express the result in rectangular form.

12. $\left[3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)\right]^3$

13. $\left[2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\right]^5$

14. $(-2 + 2\boldsymbol{i})^3$

15. $\left(1 + \sqrt{3}\boldsymbol{i}\right)^4$

16. $(3 - 6\boldsymbol{i})^4$

17. $(2 + 3\boldsymbol{i})^{-2}$

18. Raise $2 + 4\boldsymbol{i}$ to the fourth power.

Find each principal root. Express the result in the form *a* + *bi* with *a* and *b* rounded to the nearest hundredth.

19. $\left[32\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)\right]^{\frac{1}{5}}$

20. $(-1)^{\frac{1}{4}}$

21. $(-2 + \boldsymbol{i})^{\frac{1}{4}}$

22. $(4 - \boldsymbol{i})^{\frac{1}{3}}$

23. $(2 + 2\boldsymbol{i})^{\frac{1}{3}}$

24. $(-1 - \boldsymbol{i})^{\frac{1}{4}}$

25. Find the principal square root of $\boldsymbol{i}$.

Solve each equation. Then graph the roots in the complex plane.

26. $x^3 - 1 = 0$

27. $x^5 + 1 = 0$

28. $2x^4 - 128 = 0$

29. $3x^4 + 48 = 0$

30. $x^4 - (1 + \boldsymbol{i}) = 0$

31. $2x^4 + 2 + 2\sqrt{3}\boldsymbol{i} = 0$

Graphing Calculator

Use a graphing calculator to find all of the indicated roots.

32. fifth roots of $10 - 9\boldsymbol{i}$

33. sixth roots of $2 + 4\boldsymbol{i}$

34. eighth roots of $36 + 20\boldsymbol{i}$

Applications and Problem Solving

Real World Application

35. Fractals Is the number $\frac{1}{2} + \frac{3}{4}\boldsymbol{i}$ in the escape set or the prisoner set for the function $f(z) = z^2$? Explain.

36. Critical Thinking Suppose $w = a + b\boldsymbol{i}$ is one of the 31st roots of 1.

a. What is the maximum value of a?

b. What is the maximum value of b?

37. **Design** Gloribel works for an advertising agency. She wants to incorporate a hexagon design into the artwork for one of her proposals. She knows that she can locate the vertices of a regular hexagon by graphing the solutions to the equation $x^6 - 1 = 0$ in the complex plane. What are the solutions to this equation?

38. **Computer Graphics** Computer programmers can use complex numbers and the complex plane to implement geometric transformations. If a programmer starts with a square with vertices at (2, 2), (−2, 2), (−2, −2), and (2, −2), each of the vertices can be stored as a complex number in polar form. Complex number multiplication can be used to rotate the square 45° counterclockwise and dilate it so that the new vertices lie at the midpoints of the sides of the original square.
 a. What complex number should the programmer multiply by to produce this transformation?
 b. What happens if the original vertices are multiplied by the square of your answer to part a?

39. **Critical Thinking** Explain why the sum of the imaginary parts of the p distinct pth roots of any positive real number must be zero.

Mixed Review

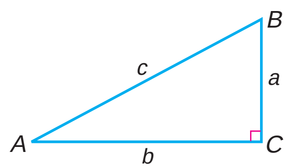
40. Find the product $2\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)\cdot 3\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right)$. Express the result in rectangular form. *(Lesson 9-7)*

41. Simplify $(2 - 5\boldsymbol{i}) + (-3 + 6\boldsymbol{i}) - (-6 + 2\boldsymbol{i})$. *(Lesson 9-5)*

42. Write parametric equations of the line with equation $y = -2x + 7$. *(Lesson 8-6)*

43. Use a half-angle identity to find the exact value of $\cos 22.5°$. *(Lesson 7-4)*

44. Solve triangle ABC if $A = 81°15'$ and $b = 28$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-4)*



45. **Manufacturing** The Precious Animal Company must produce at least 300 large stuffed bears and 400 small stuffed bears per day. At most, the company can produce a total of 1200 bears per day. The profit for each large bear is \$9.00, and the profit for each small bear is \$5.00. How many of each type of bear should be produced each day to maximize profit? *(Lesson 2-7)*

46. **SAT/ACT Practice** Six quarts of a 20% solution of alcohol in water are mixed with 4 quarts of a 60% solution of alcohol in water. The alcoholic strength of the mixture is

A 36% **B** 40% **C** 48% **D** 60% **E** 80%

Extra Practice See p. A43.